MAT1322
Lecture 10

Series

13/02/2017

<u>Review</u>: · infinite sums $\sum_{i=1}^{\infty} a_i$, eg. $\sum_{n=1}^{\infty} (\frac{1}{2})^n$

· we defined <u>partial sums</u>: $S_n = \sum_{i=1}^{n} a_i$

A series <u>converges</u> if the sequence of partial sums converge
ie. the sum $\sum_{n=1}^{\infty} a_n$ is finite!

A series <u>diverges</u> if the sequence of partial sums diverge

<u>Examples</u>

convergent series: $\sum_{n=1}^{\infty} (\frac{1}{2})^n$, $\sum_{n=1}^{\infty} q^n$, $|q|<1$

divergent series: $\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} (-1)^n$

↑
harmonic series

Geometric series $\sum_{n=1}^{\infty} q^n$ only converges if and only if $|q|<1$

$$\boxed{\sum_{n=1}^{\infty} q^n = \frac{1}{1-q}}$$ Geometric series

<u>Ex</u> Does $\sum_{n=1}^{\infty} \frac{5}{3^n}$ converge?

$= 5 \cdot \sum_{n=1}^{\infty} \frac{1}{3^n} = 5 \cdot \sum_{n=1}^{\infty} (\frac{1}{3})^n$, so it looks like a geometric series with $q = \frac{1}{3}$

$= 5 \cdot \frac{1}{1-\frac{1}{3}} = 5 \cdot \frac{3}{2} = \underline{\frac{15}{2}}$

Often we just need to know whether a series converges or diverges, not its value.

Theorem $\sum_{n=1}^{\infty} a_n$ can only converge if $\lim_{n\to\infty} a_n = 0$

This is a necessary condition, but not sufficient. In other words, just because $\lim_{n\to\infty} a_n = 0$ does not guarantee that the series converges.

Example of failure: $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series

It satisfies $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ is infinite

Theorem about series properties

If both $\sum_{n=1}^{\infty} a_i$ and $\sum_{n=1}^{\infty} b_i$ CONVERGE, so do:

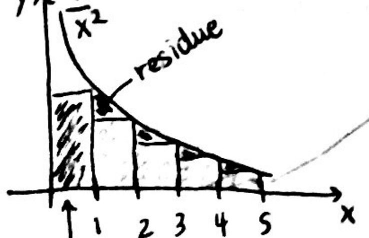
(i) $\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

INTEGRAL TEST

Use knowledge about improper integrals to investigate if a series converges or diverges.

Ex $\sum_{n=1}^{\infty} \frac{1}{n^2}$ looks like $\frac{1}{x^2}$



$\int_1^{\infty} \frac{1}{x^2}\,dx$ (area under curve)

So the sum $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$

Recall that $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges, so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ does too

INTEGRAL TEST

f is continuous, positive, and decreasing on $[1,\infty)$ and $a_n = f(n)$

i) If $\int_1^\infty f(x)\,dx$ is converging, then $\sum_{n=1}^\infty a_n$ converges.

ii) If $\int_1^\infty f(x)\,dx$ diverges, so does $\sum_{n=1}^\infty a_n$

(works only if you can determine whether $\int_1^\infty f(x)\,dx$ converges/diverges)

Ex $\sum_{n=1}^\infty \frac{1}{n^2+1}$

$f(x) = \frac{1}{x^2+1}$, compute $\int_1^\infty \frac{1}{x^2+1}\,dx$

$$\lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} (\arctan(x))\Big|_1^t = \lim_{t\to\infty} (\underbrace{\arctan(t)}_{\pi/2} - \underbrace{\arctan(1)}_{\pi/4})$$

$$= \pi/2 - \pi/4$$

$$\boxed{= \pi/4}$$

the integral converges, so $\sum_{n=1}^\infty \frac{1}{n^2+1}$ does too!

Ex $\sum_{n=1}^\infty n^2 \cdot e^{-n^3}$

$f(x) = x^2 \cdot e^{-x^3}$

$\int_1^\infty x^2 \cdot e^{-x^3}\,dx$ | $u = -x^3$, $du = -3x^2\,dx$

$$= \int_1^\infty \cancel{x^2} \cdot e^{-x^3} \cdot \frac{du}{-3\cancel{x^2}}$$

$$= \frac{-1}{3}\int_{-1}^{-\infty} e^u\,du$$

$$= \lim_{t\to\infty} \int_1^t e^u\,du = \lim_{t\to\infty} (e^{t} - e^{-1})$$

$$= -\tfrac{1}{3}\left(\tfrac{1}{e}\right) \boxed{= \frac{1}{3e}}$$

$\therefore$ converges!

COMPARISON THEOREM

Compare a series to some series we know converges or diverges.
If you have $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ and we know that $a_i \le b_i$,
and $\sum_{i=1}^{\infty} b_i$ converges, then $\sum_{i=1}^{\infty} a_i$ does (if $a_i, b_i \ge 0$).

<u>Ex</u> $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$, does it converge or diverge?

$\frac{1}{n^3+1} \le \frac{1}{n^2+1}$, so compare to $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ → from earlier, we know this converges

$\sum_{n=1}^{\infty} \frac{1}{n^3+1} \le \sum_{n=1}^{\infty} \frac{1}{n^2+1}$, so it converges too.

<u>Ex</u> $\sum_{n=1}^{\infty} \frac{1}{n!}$ $(n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)n$

$a_n = \frac{1}{n!}$, compare to $\frac{1}{(n-1)\cdot n} \le \frac{1}{n \cdot n} = \frac{1}{n^2}$

$\sum_{n=2}^{\infty} \frac{1}{n!} \le \sum_{n=2}^{\infty} \frac{1}{(n-1)n} \le 2 \sum_{n=2}^{\infty} \frac{1}{n^2}$, which converges (from integral test before)

$\boxed{\text{In general: show with the integral test that } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if and only if } p > 1.}$